

# WaveStat – Cluster Analysis of Image Data and Wavelet Coefficients

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**Abstract.** The *WaveStat* project deals with the exploratory Cluster Analysis of picture data and wavelet-built coefficients. Cluster Analysis means to collect elements (i.e. pixels) of the same property (in this context this means: gray value) to a cluster. These clusters can be visualized by colorizing them or by reconstructing them to a grayscale picture.

In this article we will show that a cluster analysis can be done directly on picture data (pixels) or on wavelet coefficients resulting from a wavelet analysis of the image. The clustered coefficients then are transformed back using the inverse wavelet transformation. The quality of this reconstruction depends on the used wavelet-transform, the used cluster algorithm and the number of iterations when performing the inverse wavelet-transform.

This work uses the redundancy of wavelet coefficients to apply cluster analysis on these to achieve an image compression.

**Keywords:** wavelet transform, cluster analysis of image data, redundancy of wavelet coefficients

## 1 Introduction

Publications about wavelets are including a broad spectrum of themes like the development of different wavelet types, their applications on different types of images, the boundary value solution problem and their mathematical properties [4, 8, 10, 11, 12]. This kind of work is mainly concerned with one effect of the multiresolution property: the redundancy of wavelet coefficients and the attempt to reduce it by the exploratory analysis of similarities between them.

Similarity analyses can be conducted by heuristic classification approaches which are also used for pattern recognition of signals. Wavelet coefficients themselves can also be interpreted as signals which can be analysed.

## 2 Wavelets

Let us recall the notions of multiresolution analysis briefly.

Wavelet transforms are integral transforms allowing a multiresolution analysis by variations of  $\Delta t$  and  $\Delta f$  in the time-frequency domain.

The two-dimensional continuous wavelet transform of a function  $f \in L^2(\mathbb{R})$  with the basis

wavelet  $\psi$  is defined by  $L_\psi f(a, b) = \frac{1}{\sqrt{c_\psi}} |a|^{-1/2} \int_{\mathbb{R}} f(t) \psi\left(\frac{t-b}{a}\right) dt$ . [10]

A signal can be analysed by a wavelet basis function  $\psi$  constituting  $L^2$  scalar products of  $f$  with dilated and translated versions of  $\psi$ .  $\tilde{L}_\psi f(a, b) = |a|^{-1/2} \int_{\mathbf{R}} f(t) \psi\left(\frac{t-b}{a}\right) dt$ . [5]

The basis  $\psi$  can be derived from a prototype corresponding to every type of bandpass function scaled with a factor  $a$  and translated with a parameter  $b$  and fulfilling the condition

$$0 < c_\psi := 2\pi \int_{\mathbf{R}} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty. [3]$$

If the parameter  $b$  was to be shifted for a wavelet then  $L_\psi f(a, b)$  contained local information about  $f$  for  $t = b$ . The extent of the domain of the wavelet is depending on the scaling parameter  $a$ . If  $a$  decreases to zero the wavelet transform will be localized on  $t = b$ .

The collection of coefficients for all  $a$  and  $b$  is highly redundant. The continuous wavelet transform allows all values of  $a$  and  $b$ .

A higher time resolution and a lower frequency resolution are achieved for signals with a lower frequency and a lower time resolution and a higher frequency resolution for signals with a higher frequency.

The wavelet transform is isometric in Hilbert spaces like the Fourier transform so that  $L_\psi f$  can be interpreted as a folding of  $f$  with a dilated wavelet  $\psi(\bullet/a)$  for every firm parameter value of  $a$ .

The partial isometry of wavelet transform in Hilbert spaces serves as the underlying basis for the inversion of the wavelet transform on its image domain by the adjuncted image  $L_\psi^*$  [4,10].

The adjuncted operator  $L_\psi^* L^2\left(\mathbf{R}^2, \frac{dad b}{a^2}\right) \rightarrow L^2(\mathbf{R})$ ,  $g \rightarrow c_\psi^{-1/2} \int_{\mathbf{R}} \int_{\mathbf{R}} |a|^{-1/2} \psi\left(\frac{t-b}{a}\right) g(a, b) \frac{dad b}{a^2}$

inverts the wavelet transform on the image domain. The signal reconstruction is defined by:

$$\left\langle \tilde{L}_\psi f_1, \tilde{L}_\psi f_2 \right\rangle_{L^2(\mathbf{R}^2, dad b/a^2)} = \underbrace{2\pi \int_{\mathbf{R}} \frac{\hat{\psi}(\omega) \overline{\hat{\psi}(\omega)}}{|\omega|} d\omega}_{=: c_{\psi\psi}} \langle f_1, f_2 \rangle_{L^2}, \quad c_{\psi\psi}^{-1} \tilde{L}_\psi^* \tilde{L}_\psi f = f. [9, 16]$$

## 2.1 Multiresolution analysis and the pyramidal algorithm

For the most compact representation in signal processing orthonormal wavelet bases are used which are non-redundant and compact and which can be constructed by choosing discrete values of  $a$  and  $b$ .

In an orthonormal multi-resolution analysis the Hilbert space  $L^2$  is sequentially divided up in subordinated closed spaces  $V_i$ , each of them corresponding to one scale defined by the scaling function  $\phi$  or by the scaling filter of the pyramidal algorithm [12].

$L^2(\mathbb{R}^2)$  is a vector space for all measurable quadratic integrable one-dimensional functions  $f(x)$  and  $A_{2^j}$  is a linear operator defining an orthonormal projection for one vector space  $V \in L^2(\mathbb{R})$ .

For the implementation of WaveStat the discretized form related to parameter  $a$  and  $b$  was

$$\text{used: } f(x) = \frac{1}{\sqrt{c_\psi}} \iint_{\mathbb{R} \times \mathbb{R}} L_\psi f(a, b) \frac{1}{\sqrt{|a|}} \psi\left(\frac{x-b}{a}\right) \frac{dad b}{a^2} \text{ with } a = 2^j \text{ and } j = 1, 2, 3 \text{ and } b = 1, 2, 3 \text{ so}$$

that a most compact representation in signal processing was achieved.

Let  $A_{2^j} f(x)$  be the approximation of  $f(x)$  with a resolution of  $2^j$  then  $V_{2^j}$  can be described as a set of all possible approximations of functions in  $L^2(\mathbb{R}^2)$ . Each information about the signal approximation with the resolution of  $2^j$  is included in the approximation with the next higher degree of resolution  $2^{j+1}$ .

The discrete approximation  $A_{2^j}^d f = \left( \langle f(u), \phi_{2^j}, (u - 2^j n) \rangle \right)_{n \in \mathbb{Z}}$  of  $f(x)$  with a resolution of  $2^j$  can be interpreted as a lowpass filtering of  $f(x)$  by suppression of the higher frequencies of  $f(x)$ . The extraction of the information difference between two different levels of resolutions  $2^j$  and  $2^{j+1}$  is achieved by the orthogonal projection of the original signal on the orthogonal complement  $O_{2^j}$  of  $V_{2^j}$  in  $V_{2^{j+1}}$  and will provide the highpass filtered signal. The orthonormal basis function of  $O_{2^j}$  has to be determined by scaling and shifting of a basis wavelet function  $\psi(x)$  which can be interpreted as a bandpass filter. The multiresolution non standard wavelet analysis was implemented with a pyramid algorithm which samples  $A_{2^{j+1}}^d f$  with two discrete quadratic mirror filter pairs  $G(\omega)$  (detail filter) and  $H(\omega)$  (lowpass filter). Each filtering process of the pyramid algorithm reduces iteratively the signal to the half and decomposes it in a detail signal  $D_{2^j} f$  and a lowpass signal  $A_1^d f$ . The one-dimensional case is easily to generalize for higher dimensions [7, 15].

A two dimensional image  $f(x, y) \in L^2(\mathbb{R}^2)$  with  $V_{2^{j+1}}^2$  will be wavelet tranformed and in the inverse wavelet transform reconstructed by shifted and dilated versions of a scaling function  $\phi$  and the wavelet basis functions  $\psi^1, \psi^2, \psi^3$ .

In the non-standard pyramid algorithm for the two-dimensional case in each processing step a reduction of the signal about factor 4 is achieved with a lowpass filter  $h(x) \otimes h(y)$ , a vertical highpass filter  $h(x) \otimes g(y)$ , a horizontal high pass filter  $g(x) \otimes h(y)$  and a diagonal highpass filter  $g(x) \otimes g(y)$ .

In the implementation of the standard wavelet transform first a one-dimensional wavelet transform is calculated for the gray values of each pixel row so that an average value of all detail coefficients will be obtained. Second, a one-dimensional wavelet analysis is applied to the rows of coefficients. The resulting coefficient are equivalent to detail coefficients with the exception an average coefficient value.

Following wavelet basis functions were implemented in a standard and in a non-standard transform: Adelson-, Battle-Lemarie, Coiflet-, Pseudo-Coiflets, Daubechies-, Linear-, Haar-, cubic and linear B-spline wavelets [6].

### **3 Cluster Analysis**

For classification we used a new optimized method of cluster analysis. First, we want to introduce basic principles of cluster analysis and then to give an overview of the methods that were implemented in WaveStat.

#### **3.1 Basic principles**

Cluster analysis means a classification procedure for identifying homogeneous subsets of objects from a heterogeneous entity of objects using a measure of distance. Cluster analyzing methods are using heuristic algorithms that build classifications and hierarchies with properties dependent on the implementation of the algorithm. Originally developed for applications in psychological and social sciences and not for the application of pixel data sets there is very short literature about cluster analysis as image operator [13].

In contrast to other uni- or bivariate types of analyses the data given by cluster analysis consist of a multitude of variables or objects or picture elements. The aim of a cluster analysis is the concentration of objects to groups given by a similarity between them. Between the different groups should be less similarity than within the groups. There are two main steps in a cluster analysis:

- Selection of a measure of proximity (making of distance matrix): the extent of proximity for two objects each is being measured and expressed by a numerical value,

- Selection of a fusion algorithm: based on their proximities the objects are grouped to clusters in a way that distinguishes them from other clusters. In further steps clusters are being clustered iteratively thus building a hierarchy of clusters.

Depending on the data niveau (nominal-, ordinal- or interval-scaled) one has to choose between different proximity/distance – concepts. Grayscaled image elements are interval-scaled.

### 3.1.1 Measures of similarity and distance

Similarity or distance of objects can be defined only on such object properties that can be measured for all objects. *Similarity* means a function  $s \in \mathbf{R} : E \times E \rightarrow \mathbf{R}$ , distributing two elements  $e_i, e_j$  a number  $s = s(e_i, e_j) = s_{ij} \in \mathbf{R}$ . ( $E$  is the set of all elements).  $s$  is within an interval  $s_0 \leq s \leq s_1$ ,  $s_{ij} = s_1$  means maximal,  $s_{ij} = s_0$  means minimal similarity. Values  $s_{ij}$  are stored in a (symmetric) similarity matrix  $\mathbf{S} = (s_{ij})$ . A function  $d \in \mathbf{R} : E \times E \rightarrow \mathbf{R}$ , distributing a non-negative number  $d = d(e_i, e_j) = d_{ij} \in \mathbf{R}$  is called *distance*. Here too, values  $d_{ij}$  are stored in a symmetric  $(n, n)$ -matrix  $\mathbf{D} = (d_{ij})$ .  $\mathbf{D}$ 's diagonal elements equal to 0. This allows to save much space because there are  $n(n-1)/2$  independent values left to store in memory.

Similarity and distance can be transformed into each other; their concepts are identical to cluster analysis. For a given  $r \in \mathbf{Z}$  the distance of two elements  $e_i, e_j \in E$  can be expressed by

Minkowski-r-metrics:  $d_r(x_i, x_j) := \|x_i - x_j\|_r := \sqrt[r]{\sum_{l=1}^m |x_{il} - x_{jl}|^r}$ .  $x'_i = (x_{i1}, \dots, x_{im})$  and

$x'_j = (x_{j1}, \dots, x_{jm})$  are vectors representing elements  $e_i, e_j$  in  $\mathbf{R}^m$ , given by quantitative properties like grayscale-values of an image.  $r=1$  gives a *city-block metric*,  $r=2$  results in the well-known euclidian distance. (See [2,14].)

### 3.1.2 Cluster analysis procedures

A classification of cluster analysis methods can be made with regard to their grouping results, the process of clusterization and the clusterization criterions. The methods used in WaveStat are hierarchical and *agglomerative*, that means that single elements are agglomerated to bigger clusters in contrast to *divisive* methods where the starting point is one big cluster that has to be divided into smaller clusters. Agglomerative methods consist of five main steps [14]:

1. Start with the smallest partition  $G := \{g_1, \dots, g_n\}$ , with  $g_i = \{e_i\}$ .

2. Look for groups that have minimal distance, that means groups  $g_p$  and  $g_q$  with

$$d_{pq} = \min_{i \neq j} d_{ij}.$$

3. Merge groups  $g_p$  and  $g_q$  to a new group  $g_q^{new}$  thus reducing the amount of all groups by 1.

4. Change row and column  $q$  of the distance matrix by recalculating the distances between  $g_q^{new}$  and all other groups. Erase row and column  $p$  of the distance matrix.

5. After  $n-1$  steps (all groups are merged to a single group now) end; otherwise continue at step 2 with the changed distance matrix.

Most differences in agglomerative methods can be found in step 4, where the distance between the new and the the other groups is being calculated. In some cases, the defined grouping boundaries encounter some modifications: if remote elements are being added to a cluster the cluster space is contracted (contraction effect). A large amount of small and very small groups results in a stretching of the cluster space (dilatation effect). Conservative methods try to avoid such effects. The used methods in WaveStat are: Single, Average, Weighted Average and Complete Linkage, Median, Centroid and Ward's Method and Flexible Strategy. Their distance metrics and properties are being described in the following table [14].

Method	Distance metric	Properties / Remarks ( $n:=n_p+n_q$ )
Single Linkage (Nearest Neighbor)	$\min(d_{pi}, d_{qi})$	Contracting, monotone, invariant with monotone transformation of the distance matrix; building of chains
Complete Linkage (Furthest Neighbor)	$\max(d_{pi}, d_{qi})$	dilatating, monotone, invariant with monotone transformation of the distance matrix; tendency of small groups
Average Linkage	$\frac{1}{2}(d_{pi} + d_{qi})$	conservative, monotone
Weighted Average Linkage	$\frac{1}{n}(n_p d_{pi} + n_q d_{qi})$	conservative, monotone, weighted with number of elements
Median	$\frac{1}{2}(d_{pi} + d_{qi}) - \frac{1}{4}d_{pq}$	Conservative, not monotone (inversions), for squared euclidian distances only
Centroid	$\frac{1}{n}(n_p d_{pi} + n_q d_{qi}) - \frac{n_p n_q}{n^2} d_{pq}$	Conservative, not monotone (inversions), not for squared euclidian distances
Ward	$\frac{1}{n + n_i}((n_i + n_p)d_{pi} + (n_i + n_q)d_{qi} - n_i d_{pq})$	Conservative, monotone, minimization of variance criterion with squared euclidian distances
Flexible Strategy	$\alpha(d_{pi} + d_{qi}) + (1 - 2\alpha)d_{pq}$	$\alpha \rightarrow 0$ : contracting, $\alpha = 1/2$ : conservative, $\alpha > 1/2$ : dilatating. optimal: $0.6 \leq \alpha \leq 0.7$

Tab.1: Clusterization strategies

#### 4 WaveStat

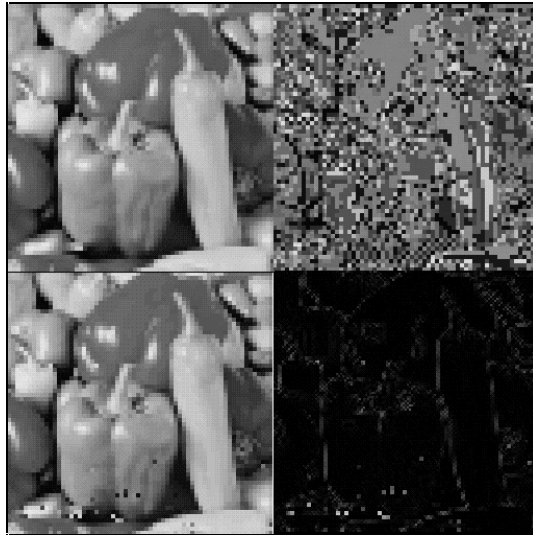
WaveStat is a software running under X-Windows (portation to other OS's would be simple) that provides all functions and procedures mentioned above [1]. The menu-driven user

interface is simple and functional, all operations are selectable in the main dialog. All examined images are of square shape for simplicity and performance reasons; they all are grayscale. In principle, an extension to coloured, non-square pictures is possible and planned for future work. WaveStat reads raw, binary and pgm-coded grayscale images.

#### **4.1 Wavelet and clustering types**

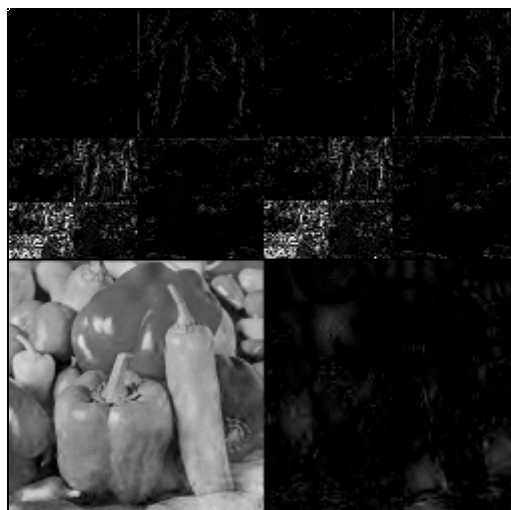
After loading an image the user is allowed to perform a wavelet transform on the image data. All wavelet types, as mentioned above, may be performed as standard and non-standard transform. Selecting a standard transform results in displaying new windows on the display showing original image, wavelet coefficients and the result of the inverse wavelet transform. The cluster algorithms correspond to the methods described above. Further steps are dependent on the data the user will cluster and the chosen wavelet transform if one selects the cluster analysis for wavelet coefficients and not for the original image. According to the used method of image processing WaveStat provides additional data visualization. The ,standard‘ cluster algorithms provide a linewise clustering of the data and the results to be visualized as dendrograms. For WaveStat, we implemented another clustering method that calculates distances between each pixel of the image and not only between the single lines of it. This type of clustering has been called ,enhanced‘ clustering. Since dendrograms used in classical cluster analysis doesn‘t lead to appropriate information about the cluster distribution for an image or its wavelet coefficients a different visualization method to illustrate enhanced clustering had to be implemented: elements within the same cluster get the same colour. This results in a new image of the clusters showing colourized element groups reflecting the gray value boundaries within the original image. In a second step the average gray value for each cluster was calculated and visualized as a coloured image pattern again. The similarity between original image and the image of the averaged cluster values leads to a qualitative criterion for the goodness of reconstruction for the used cluster algorithm.

If the user choses the clustering of the original image data new windows will appear on the screen showing a graphical representation of the cluster and a transform from the cluster data back to a grayscale picture. Additional, difference information between original image and back-translated picture is provided in another window. Fig. 1 shows (first row) the original image, visualization of the cluster data, (second row) the grayscale image translated back from the cluster information and the differences between original and resulting image.



**Fig. 1: Clustering of original image data**

Selecting the clustering of wavelet coefficients leads to different results depending on the transform type: clustering standard wavelet transform coefficients shows the original image, images of coefficients and their cluster visualization and, finally, the results of the inverse wavelet transform made on the clustered and averaged coefficients. A cluster analysis of nonstandard wavelet coefficients takes iteratively the coefficients of only one resolution level and assembles the results – three cluster images per degree of resolution – into one new window. Then an inverse wavelet transform is done on this new picture. Again, differences between this new and the original image are displayed in another window.



**Fig. 2: Clustering nonstandard wavelet coefficients**

Fig. 2 shows (first row) the original wavelet coefficients (cubic spline), their grayscaled cluster visualization, (second row) reconstruction from cluster visualization, difference between reconstruction and original image (original is the same as in Fig. 1).

WaveStat allows a lot of additional settings: the z-transform-parameter can be set to do a z-transform on the data before computing the distance matrix. A selection between ,normal‘ (linewise) and ,enhanced‘ clustering determines calculation and visualization of the clustered data. Linewise clustering shows a dendrogram, data are not reconstructed; enhanced clustering builds a distance-matrix that stores distances between each pairs of pixels of the image and arranges for the reconstruction of the clustered data. The amount of resolution levels that is used by reconstruction of clustered nonstandard wavelet coefficients can be set with the Levels-parameter. Also some different methods for a downscaling of the entry matrix and pre-clustering optimization strategies with adaptive tolerance levels are provided to accelerate image processing by reduction of the distance matrix. After the computation of the distance matrix equal or similar gray values can be condensed depending on the chosen tolerance level.

For the evaluation of our program we used different photorealistic and simple geometric images.

## 5 Results

Giving the complete results of all possible combinations between different modifications in detail would lead too far so we concentrate on some main results.

For an evaluation of the results following factors are of importance:

- the wavelet function type used for the analysis and the inverse form for the reconstruction
- the cluster algorithm (optimized / not optimized, clustering of the gray value matrix of the original image or the wavelet coefficients)
- the parameter values ,levels‘ (only for the cluster analysis of wavelet coefficients) and ,tolerance‘ (cluster analysis of the original images and wavelet coefficients).

In general the results obtained with WaveStat can be divided up into two main groups: the results of the cluster analyses of the whole image value matrix and the results of the examination of wavelet coefficients.

A combination between those main procedures provides the analysis of standard transform wavelet coefficients. A division into subsets of wavelet coefficients according to the levels of iteration of the pyramid algorithm and a single cluster analysis of the coefficients of each

iteration level is not possible so that the coefficients of the whole image have to be processed in one cluster analysis.

## **5.1 The influence of the cluster method**

The type of cluster algorithm is a relevant factor for the analysis of nonstandard wavelet coefficients and the clustering of the original image. For the clustering of standard wavelet coefficients the obtained results show a minor importance because reconstruction artifacts are mainly caused by an interaction effect with the method of implementation of the wavelet transform.

The best results for all types of images were achieved with the single linkage and the complete linkage method in combination with the optimization. By the introduction of a threshold or tolerance value the optimization strategy of the original cluster algorithm is changed with a kind of pre-clustering step. Methods like average linkage or flexible strategy have proven not being suitable for this additional optimization strategy.

For the non-optimized clustering strategies there are no qualitative differences in the clustering of the original images while for the clustering of coefficients the weighted average linkage and the flexible strategy method have shown qualitative not so good results.

For all methods which have been shown of being appropriate for optimization strategies there are no qualitative differences neither for the cluster analysis of original images nor for the clustering of wavelet coefficients.

The choice between the euclidian distance metric and the city block metric doesn't lead to any differences in the reconstructed images although the quadratic euclidian distance metric enhances contrasts by higher weighting of larger differences between gray values or wavelet coefficients while a city block metric allows an equal weighting between larger and smaller differences of grey values because our sample images don't show very large leaps in their gray value distributions.

## **5.2 Results of the reconstruction of standard transform wavelet coefficients**

The results of the cluster analysis of the standard transform wavelet coefficients could be neglected because during the inverse wavelet image reconstruction the resulting new clustered and averaged wavelet coefficients were processed as a whole and not separately as filter types and levels of iterations.

The developed artifacts of the reconstructed images were different depending on the type of wavelet function. Spline wavelet coefficients were not suitable for any reconstruction. Because of their biorthogonal properties it is necessary to use different filter functions for the analysis and the reconstruction of an image.

### **5.3 Results of the reconstruction of non standard transform wavelet coefficients**

The quality of image reconstruction was less dependent on the type of clustering method or optimization methods but more influenced by the wavelet function type and the chosen values for the parameters ,level‘ and ,tolerance‘.

Generally the image reconstruction from clustered and averaged wavelets received from non-standard wavelet analysis algorithms has proven a high quality compared to the results for standard transform algorithms.

Some artifacts in the reconstruction, however, could be noticed according to the chosen wavelet basis function. For Daubechies wavelet basis functions there were wave-shaped irregularities in the reconstructed image increasing with the number of vanishing moments, reconstructions based on Haar wavelet coefficients showed artifacts in rectangular forms but without any blurring of contours or loss of edge detection. Coiflet and Pseudo coiflet reconstructions produced some fissures within the presented image structures without any loss in edge detection. Cubic spline wavelet analysis and reconstruction were both possible in the nonstandard implementation.

## **6 Conclusion**

WaveStat has shown some basic facts:

- clustering image data is possible and sensible; for image data as well as wavelet coefficients
- cluster analysis has to be modified to guarantee good results: the distance matrix must not be built the conventional way but differences have to be computed for each pair of pixels. This will raise the cost of computation significantly, so the algorithm has to be improved.
- The procedure does not work correctly on standard wavelet coefficients.
- WaveStat may be a good starting point for classification procedures as well as a new image compression method which works very effectively.

Further versions of WaveStat will concentrate on the improvement of the clustering process, especially the building of the distance matrix and the compression of the colour space for coloured images.

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